On the Maximum Drawdown of a Brownian Motion

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Abstract

If $X(t)$ is a random process on $[0, T]$, the maximum drawdown at time T, $\bar{D}(T)$, is defined by

$$\bar{D}(T) = \sup_{t\in[0,T]} \left[\sup_{s\in[0,t]} X(s) - X(t) \right].$$

Informally, this is the largest drop from a peak to a bottom. In this paper, we investigate the behavior of this statistic for a Brownian motion with drift. In particular, we give an infinite series representation of its distribution, and consider its expected value. When the drift is zero, we give an analytic expression for the expected value, and for non-zero drift, we give an infinite series representation. For all cases, we compute the limiting ($T \to \infty$) behavior, which can be logarithmic ($\mu > 0$), square root ($\mu = 0$), or linear ($\mu < 0$).

Keywords: Random Walk, Asymptotic Distribution, Expected Maximum Drawdown.

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1 Introduction

The maximum drawdown is a commonly used in finance as a measure of risk for a stock that follows a particular random process. Here we consider the maximum drawdown of a Brownian motion. Let $W(t)$, $0 \leq t \leq T$, be a standard Wiener process, and let $X(t)$ be the Brownian motion given by $X(t) = \sigma W(t) + \mu t$, where $\mu \in \mathbf{R}$ is the drift and $\sigma \geq 0$ is the diffusion parameter. The maximum drawdown is defined by

$$\bar{D}(T; \mu, \sigma) = \sup_{t \in [0,T]} \left[\sup_{s \in [0,t]} X(s) - X(t) \right]. \tag{1}$$

(we will supress the T, μ, σ dependence). Denote the distribution function for $\bar{D}$ by $G_{\bar{D}}(h) = P[\bar{D} \geq h]$. We find that

$$G_{\bar{D}}(h) = 2\sigma^4 \sum_{n=1}^{\infty} \frac{\theta_n \sin \theta_n}{(\sigma^4 \theta_n^2 + \mu^2 h^2 - \sigma^2 \mu h)} e^{-\frac{\mu h}{\sigma^2}} \left(1 - e^{-\frac{\sigma^2 \theta_n^2 T}{2h^2}} e^{-\frac{\mu^2 T}{2\sigma^2}} \right) + L, \tag{2}$$

Here, $\{\theta_n\}$ are the positive solutions of the eigenvalue condition

$$\tan \theta_n = \frac{\sigma^2}{\mu h} \theta_n, \tag{3}$$

and L is given by

$$L = \begin{cases} 0 & \mu < \dfrac{\sigma^2}{h}, \\ \dfrac{3}{e}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2}}\right) & \mu = \dfrac{\sigma^2}{h}, \\ \dfrac{2\sigma^4 \eta \sinh \eta \; e^{-\frac{\mu h}{\sigma^2}}}{(\sigma^4\eta^2 - \mu^2 h^2 + \sigma^2 \mu h)} \left(1 - e^{-\frac{\mu^2 T}{2\sigma^2}} e^{\frac{\sigma^2 \eta^2 T}{2h^2}}\right) & \mu > \dfrac{\sigma^2}{h}, \end{cases} \tag{4}$$

where η is the unique positive solution of

$$\tanh \eta = \frac{\sigma^2}{\mu h}\eta. \tag{5}$$

Using the identity, $E\left[\bar{D}\right] = \int_0^\infty dh \; G_{\bar{D}}(h)$ and defining $\alpha = \mu\sqrt{T/2\sigma^2}$, we find that

$$E\left[\bar{D}\right] = \frac{2\sigma^2}{\mu} Q_{\bar{D}}(\alpha^2), \tag{6}$$

where

$$Q_{\bar{D}}(x) = \begin{cases} Q_p(x) & \mu > 0, \\ \gamma\sqrt{2x} & \mu = 0, \\ -Q_n(x) & \mu < 0, \end{cases} \tag{7}$$

$\gamma = \sqrt{\pi/8} \approx 0.6267$ is a constant, and Q_p, Q_n are functions whose exact expressions are given in (19) and (28). These functions can be numerically evaluated (a comprehensive list of values is given in appendix B), and their asymptotic behavior is given by

$$Q_p(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ \frac{1}{4}\log x + 0.49088 & x \to \infty, \end{cases} \qquad Q_n(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ x + \frac{1}{2} & x \to \infty. \end{cases} \tag{8}$$

The asymptotic behavior is logarithmic for $\mu > 0$, linear for $\mu < 0$ and square root for $\mu = 0$. A similar result in the asymptotic case was obtained by Berger and Whitt [Berger and Whitt, 1995]. For positive μ, by considering a reflected Brownian motion [Graversen and Shiryaev, 2000], they consider the asymptotic distribution of queuing processes, and obtain a Gumbel distribution, which is consistent with our findings. Douady, Shiryaev and Yor, [Douady *et al.*, 2000], consider the maximum drawdown for a Brownian motion when the drift is zero and obtain similar results to ours in this special case. The behavior of $Q_{\bar{D}}(x)$ is shown in Figure 1.

2 Derivations

Let $D(t)$ be the random process defined by

$$D(t) = \sup_{s\in[0,t]} X(s) - X(t). \tag{9}$$

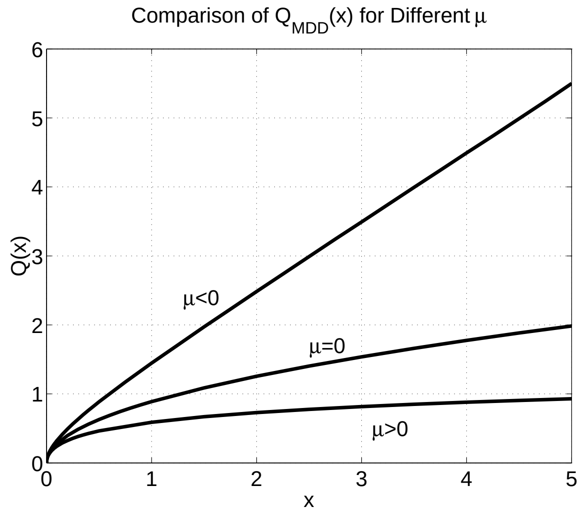

Figure 1: Behavior of the $Q(\cdot)$ functions for positive, negative and zero μ.

$D(t)$ is the drawdown from the previous maximum value at time t. It is well known that $D(t)$ is a reflected Brownian motion[1] on $[0, T]$,

$$dD(t) = \begin{cases} -dX(t) & D(t) > 0 \\ \max\{0, -dX(t)\} & D(t) = 0. \end{cases} \tag{10}$$

The rigorous justification of this fact can be found in [Graversen and Shiryaev, 2000]. From $D(t)$, we can get the maximum drawdown, $\bar{D} = \sup_{t \le T} D(t)$. Let $h > 0$ be an absorbing barrier, let τ be the absorbtion time, and let f_τ be the absorbtion time density, i.e., $f_\tau(t|h)dt$ is the probability

[1]If $X(t)$ has drift and diffusion parameters μ_X and σ, then $D(t)$ is a Brownian motion with drift $-\mu_X$ and diffusion parameter σ; $D(0) = 0$, and $D(t)$ has a reflective barrier at 0.

that $\tau \in [t, t+dt]$. Let $G_{\bar{D}}(h)$ be the probability that $\bar{D} \geq h$ in the interval $[0, T]$. Then,

$$G_{\bar{D}}(h) = \int_0^T dt \; f_\tau(t|h). \tag{11}$$

$f_\tau(t|h)$ can be computed from a more general result for a Brownian motion between two partially absorbing barriers given in [Dominé, 1996], and is given by

$$f_\tau(t|h) = e^{-\frac{\mu^2 t}{2\sigma^2}} \left[\frac{\sigma^2}{h^2} \sum_{n=0}^{\infty} \frac{(\sigma^4\theta_n^2 + \mu^2 h^2)\theta_n \sin\theta_n}{(\sigma^4\theta_n^2 + \mu^2 h^2 - \sigma^2 \mu h)} e^{-\frac{\mu h}{\sigma^2}} e^{-\frac{\sigma^2\theta_n^2 t}{2h^2}} + K \right], \tag{12}$$

where $\{\theta_n\}$ are the positive solutions to the eigenvalue condition (3), K is given by

$$K = \begin{cases} 0 & \mu < \dfrac{\sigma^2}{h}, \\[2ex] \dfrac{3\sigma^2}{2eh^2} & \mu = \dfrac{\sigma^2}{h}, \\[2ex] \dfrac{\sigma^2}{h^2} \dfrac{(\mu^2 h^2 - \sigma^4\eta^2)\eta \sinh\eta}{(\sigma^4\eta^2 - \mu^2 h^2 + \sigma^2\mu h)} e^{-\frac{\mu h}{\sigma^2}} e^{\frac{\sigma^2\eta^2 t}{2h^2}} & \mu > \dfrac{\sigma^2}{h}, \end{cases} \tag{13}$$

and η is the unique positive solution to the eigenvalue condition (5). Alternatively, in Section 3, we obtain the same result by taking the continuous limit of the discrete random walk. Using (12) in (11), and after an integration, we arrive at (2). To get the expectation, we use the identity $E\left[\bar{D}\right] = \int_0^\infty dh \; G_{\bar{D}}(h)$, which is valid for positive valued random variables.

2.1 $\mu = 0$

The eigenvalue condition (3) is solved by $\theta_n = (n - \frac{1}{2})\pi$. Thus, we have that

$$G_{\bar{D}}(h) = 2\sum_{n=1}^{\infty} \frac{\sin(n-\frac{1}{2})\pi}{(n-\frac{1}{2})\pi}\left(1 - e^{-\frac{\sigma^2(n-\frac{1}{2})^2\pi^2 T}{2h^2}}\right), \tag{14}$$

$$= \frac{2}{\pi}\, g\left(\frac{h}{\pi\sigma\sqrt{T}}\right), \tag{15}$$

where $g(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})}\left(1 - e^{-\frac{(n+\frac{1}{2})^2}{2x^2}}\right)$. The expected value of $\bar{D}$ is then given by

$$E\left[\bar{D}\right] = \int_0^\infty dh\; G_{\bar{D}}(h) = \frac{2}{\pi}\int_0^\infty dh\; g\left(\frac{h}{\pi\sigma\sqrt{T}}\right) = 2\gamma\sigma\sqrt{T}, \tag{16}$$

where we have defined the constant $\gamma = \int\limits_0^\infty dh\;\; g(h) = \sqrt{\frac{\pi}{8}} \approx 0.6267$. The exact computation of γ from this integral is challenging. An alternate route to the same expression using reflected Brownian motion was obtained by Greg Bond, [Bond, 2003], who showed that $\gamma = \sqrt{\pi/8}$. A useful comparison is to the expected value of the range R (the difference between the two extrema of the motion), which is computed for $\mu = 0$ in [Feller, 1951], and the generalization to non-zero μ can be obtained from equation (66) in appendix A and the identity $E[R] = E[H|\mu] + E[H|-\mu]$, where H is the maximum of the motion. For $\mu = 0$, $E[R] = 2\sqrt{\frac{2}{\pi}}\sigma\sqrt{T}$, and $\sqrt{\frac{2}{\pi}} \approx 0.798 > \gamma$, thus the expected range is considerably larger than the expected maximum drawdown for $\mu = 0$.

2.1.1 $\mu < 0$

After applying the eigenvalue conditions and taking the integral of $G_{\bar{D}}(h)$ to get the expectation, we arrive at

$$E\left[\bar{D}\right] = \int_0^\infty dh\ G_{\bar{D}}(h) = 2\int_0^\infty dh\ e^{-\frac{\mu h}{\sigma^2}} \sum_{n=1}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n} \left(1 - e^{-\frac{\mu^2 T}{2\sigma^2 \cos^2\theta_n}}\right). \tag{17}$$

Making a change of variables to $u = -\mu h/\sigma^2$, we find that

$$E\left[\bar{D}\right] = -2\frac{\sigma^2}{\mu} Q_n\left(\alpha^2\right), \tag{18}$$

where, for $x > 0$, we have defined $Q_n(x)$ by

$$Q_n\left(x\right) = \int_0^\infty du\ e^{u} \sum_{n=1}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n} \left(1 - e^{-\frac{x}{\cos^2\theta_n}}\right). \tag{19}$$

Here, $\{\theta_n\}$ satisfy $\tan\theta_n = -\theta_n/u$ and $\alpha = \mu\sqrt{T/2\sigma^2}$. The numerical computation of $Q_n(x)$ is not a straightforward task. The summation in the integrand is a function of u that decreases faster than e^{-u}. Since the magnitude of the n^{th} term in the summation is approximately $1/n$, we need to take $\Omega(e^u)$ terms in the summation to make sure that the next term left out has magnitude less than the size of the sum. Thus the efficient computation of this integral is computationally non trivial. The table in Appendix B gives approximate values of $Q_n(x)$ for various values of x, computed using an extensive numerical integration. Intermediate values can be obtained using interpolation and the asymptotic behavior is discussed below.

When $x \to 0^+$, $Q_n(x) \to \gamma\sqrt{2x}$, since in this limit, we must recover the $\mu \to 0$ behavior. We get the behavior in the $x \to \infty$ limit by noting that $R \geq \bar{D} \geq -L$ where L is the low and R is the range (high minus low). Taking expectations, and using (66), we see that for all α,

$$\frac{\alpha^2}{2} + \frac{Q_R(-\alpha)}{2} \leq Q_n(\alpha^2) \leq Q_R(-\alpha). \tag{20}$$

where $Q_R(x) = \text{erf}(x)\left(\frac{1}{2} + x^2\right) + \frac{1}{\sqrt{\pi}} x e^{-x^2}$. Asymptotically, as $\alpha \to -\infty$, this yields

$$\alpha^2 + \frac{1}{4} \leq Q_n(\alpha^2) \leq \alpha^2 + \frac{1}{2}, \tag{21}$$

from which we deduce that $Q_n(x) \to x + \epsilon(x)$ where $\frac{1}{4} \leq \epsilon(x) \leq \frac{1}{2}$. We now argue that asymptoticaly $E\left[\bar{D}\right] = E\left[R\right]$ when $\mu < 0$. By definition of $\bar{D}$, when the high occurs before the low, $\bar{D} = R$, so $E\left[\bar{D}\right] \geq E\left[R|H \to L\right]P[H \to L]$, where $A \to B$ is used to denote the event "A occurs before B". Since $\mu < 0$, $P[H \to L] \to 1$ as $T \to \infty$. Considering the range, we have $E\left[R\right] = E\left[R|H \to L\right]P[H \to L] + E\left[R|L \to H\right]P[L \to H]$. $E\left[R|L \to H\right]$ is slowly growing with T, but for $\mu < 0$, $P[L \to H] \to 0$ exponentially fast, hence the second term asymptotically approaches 0, and so we conclude that $E\left[R\right] \to E\left[R|H \to L\right]$. Thus we see that asymptotically, $E\left[\bar{D}\right] \geq E\left[R\right]$, and hence that $E\left[\bar{D}\right] = E\left[R\right]$, and so $\lim_{x\to\infty}\epsilon(x) = \frac{1}{2}$. Collecting the results together, we have,

$$Q_n(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ x + \frac{1}{2} & x \to \infty. \end{cases} \tag{22}$$

The asymptotic behavior is illustrated in Figure 2 which shows the numerical computation of $Q_n(x)$

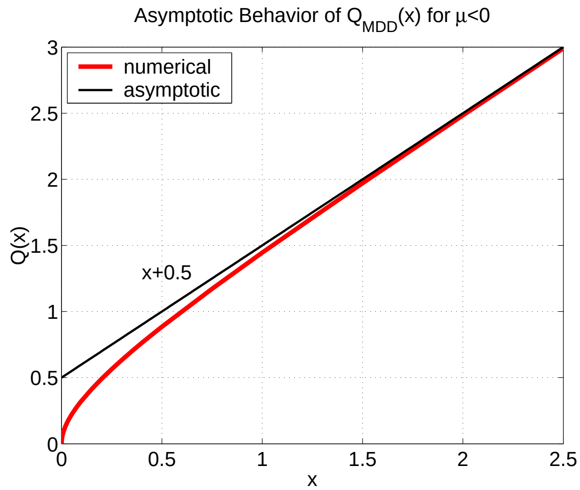

Figure 2: Asymptotic behavior of $Q_n(x)$.

along with the asymptote.

2.1.2 $\mu > 0$

In this case, for $h > \sigma^2/\mu$ in the integral, the third case for L adds another term. Thus we find that

$$
\begin{aligned}
E\left[\bar{D}\right] &= \int_0^\infty dh\, G_{\bar{D}}(h), && (23)\\
&= 2\int_0^\infty dh\, e^{-\frac{\mu h}{\sigma^2}} \sum_{n=1}^\infty \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}}\right)\\
&\quad -2\int_{\frac{\sigma^2}{\mu}}^\infty dh\, e^{-\frac{\mu h}{\sigma^2}} \frac{\sinh^3\eta}{\eta - \cosh\eta\sinh\eta}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cosh^2\eta}}\right). && (24)
\end{aligned}
$$

The second integral can be reduced by a change of variables $u = \eta(h)$ as follows. Since $\tanh u = \sigma^2 u/\mu h$ we find that

$$\frac{dh}{du} = \frac{\sigma^2}{\mu}\frac{\cosh u \sinh u - u}{\sinh^2 u}, \tag{25}$$

hence, the second integral reduces to

$$-\frac{\sigma^2}{\mu}\int_0^\infty du\, e^{-\frac{u}{\tanh u}} \sinh u \left(1 - e^{-\frac{\mu^2 T}{2\sigma^2 \cosh^2 u}}\right). \tag{26}$$

Changing variable in the first integral to $u = \mu h/\sigma^2$, we arrive at

$$E\left[\bar{D}\right] = \frac{2\sigma^2}{\mu} Q_p(\alpha^2), \tag{27}$$

where once again $\alpha = \mu\sqrt{T/2\sigma^2}$, and for $x > 0$, $Q_p(x)$ is given by

$$Q_p(x) = \int_0^\infty du \left[e^{-u}\sum_{n=0}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n}\left(1 - e^{-\frac{x}{\cos^2\theta_n}}\right) + \right.$$
$$\left. e^{-\frac{u}{\tanh u}} \sinh u \left(1 - e^{-\frac{x}{\cosh^2 u}}\right)\right]. \tag{28}$$

Here, $\{\theta_n\}$ satisfy $\tan\theta_n = \theta_n/u$. The bound (20) is still valid, but not very useful. The numerical computation of $Q_p(x)$ is relatively straightforward, as the e^{-u} term in the integrand makes it well behaved for the purposes of numerical integration. We know that $Q_p(x) \to \gamma\sqrt{2x}$ when $x \to 0^+$. We now consider the other asymptotic limit, namely $\alpha \to \infty$. We will evaluate the two contributions to $Q_p(x)$ separately. Consider the first term in (28), which we denote by $I_1(x)$,

$$I_1(x) = \int_0^\infty du\ e^{-u} \sum_{n=0}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n} \left(1 - e^{-\frac{x}{\cos^2\theta_n}}\right). \tag{29}$$

Since $0 \leq \cos^2\theta_n \leq 1$ and $x \to \infty$, the term in brackets is rapidly approaching 1. Since e^{-u} is rapidly decreasing, we interchange the summation with the integration and after changing variables in the integral to $v = \theta_n(u)$ and using the identity

$$du = \frac{\cos v \sin v - v}{\sin^2 v} dv, \tag{30}$$

we arrive at

$$I_1(x) = \sum_{n=0}^{\infty} \int_{n\pi}^{n+\frac{1}{2}\pi} dv\ e^{-\frac{v}{\tan v}} \sin v \left(1 - e^{-\frac{x}{\cos^2 v}}\right). \tag{31}$$

After translating each integral by $n\pi$ and bringing the summation back into the integral, the summation is a geometric progression which results in

$$I_1(x) = \int_0^{\frac{\pi}{2}} dv \quad \frac{e^{-\frac{v}{\tan v}} \sin v \left(1 - e^{-\frac{x}{\cos^2 v}}\right)}{1 + e^{-\frac{\pi}{\tan v}}}. \tag{32}$$

Thus, $(1 - e^{-x})\beta_1 \leq I_1(x) \leq \beta_1$ where β_1 is given by

$$\beta_1 = \int_0^{\frac{\pi}{2}} dv \quad \frac{e^{-\frac{v}{\tan v}} \sin v}{1 + e^{-\frac{\pi}{\tan v}}}, \tag{33}$$

and so we see that $I_1(x)$ rapidly converges to the constant β_1. β_1 can be evaluated numerically to give $\beta_1 = 0.4575$. Now consider the second term in (28), which we denote by by $I_2(x)$,

$$I_2(x) = \int_0^\infty du\ e^{-\frac{u}{\tanh u}} \sinh u \left(1 - e^{-\frac{x}{\cosh^2 u}}\right). \tag{34}$$

When x is large, the second term in the integrand (in parentheses) is very close to 1 until u gets large enough so that $\cosh u \sim x$, from which point this term rapidly decreases to 0. The first term in the integrand is always less than $\frac{1}{2}$ and rapidly increases from 0 to $\frac{1}{2}$. Thus we write

$$\begin{aligned} I_2(x) &= \int_0^\infty du\ \left(e^{-\frac{u}{\tanh u}} \sinh u - \frac{1}{2} + \frac{1}{2}\right)\left(1 - e^{-\frac{x}{\cosh^2 u}}\right) \\ &= \frac{1}{2}\int_0^\infty du\ \left(1 - e^{-\frac{x}{\cosh^2 u}}\right) - \int_0^\infty du\ \left(\frac{1}{2} - e^{-\frac{u}{\tanh u}} \sinh u\right) \\ &\quad + \int_0^\infty du\ e^{-\frac{x}{\cosh^2 u}} \left(\frac{1}{2} - e^{-\frac{u}{\tanh u}} \sinh u\right). \end{aligned} \tag{35}$$

We first show that the third integral approaches zero as $x \to \infty$. Since this integral is monotonically decreasing in x, it suffices to consider the integral for $x \in \mathbf{N}$. Let $g(u) = \frac{1}{2} - e^{-\frac{u}{\tanh u}} \sinh u$, and let $f_n(u) = g(u) e^{-\frac{n}{\cosh^2 u}}$. Then $|f_n| < g$ a.e. and $f_n \to 0$ a.e, therefore by the Lebesgue dominated convergence theorem, the third integral converges to 0. The second integral is a constant β_2, independent of x and can be evaluated numerically to give $\beta_2 = 0.4575$, which apparently is (numerically) equal to β_1. We suspect that $\beta_2 = \beta_1$ but the proof has been elusive.

We get bounds for the first integral using the inequalities $\cosh u \geq \frac{1}{2} e^u$, and for $u \geq A$, $\cosh u \leq \frac{1}{2} e^{\lambda(A) u}$ where $\lambda(A) = 1 + e^{-2A}/A$. Denoting the first integral in (34) by $F(x)$, we immediately get

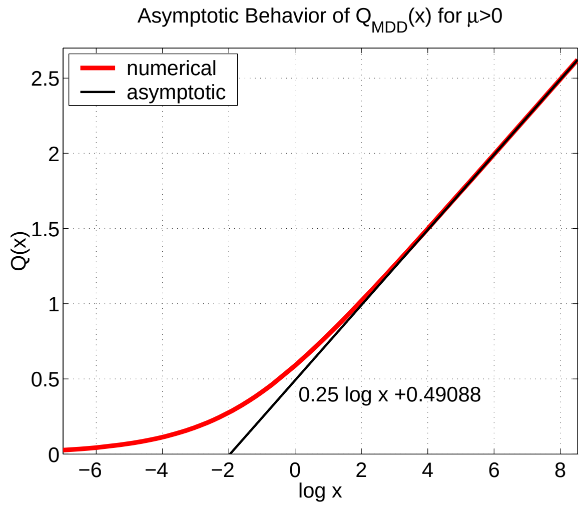

Figure 3: Asymptotic behavior of $Q_p(x)$.

the following bounds.

$$A\left(1-e^{-\frac{x}{\cosh^2 A}}\right)+\int_A^\infty du\ \left(1-e^{-4xe^{-2\lambda(A)u}}\right)\le 2F(x)\le \int_0^\infty du\ \left(1-e^{-4xe^{-2u}}\right), \tag{36}$$

which holds for arbitrary fixed A. A change of variables to $v = xe^{-2\lambda(A)u}$ in the lower bound and $v = xe^{-2u}$ in the upper bound then leads to the following bounds,

$$A\left(1-e^{-\frac{x}{\cosh^2 A}}\right)+\frac{1}{2\lambda}\int_0^{xe^{-2\lambda A}}\frac{du}{u}\left(1-e^{-4u}\right)\le 2F(x)\le \frac{1}{2}\int_0^x \frac{du}{u}\left(1-e^{-4u}\right). \tag{37}$$

We can get an asymptotic form as follows. Suppose $z > 1$, then the following identity holds,

$$\int_0^z \frac{du}{u}\left(1-e^{-4u}\right) \quad = \quad \int_0^1 \frac{du}{u}\left(1-e^{-4u}\right)+\log z-\int_1^z \frac{du}{u}e^{-4u}. \tag{38}$$

The first term is a constant, and when $z \to \infty$, the third term converges to $Ei(-4)$ (see for example [Gradshteyn and Ryzhik, 1980]). Thus, for $z \to \infty$,

$$\int_0^z \frac{du}{u}\left(1 - e^{-4u}\right) \to \log z + C, \tag{39}$$

where $C = \int_0^1 \frac{du}{u}\left(1 - e^{-4u}\right) + Ei(-4) \approx 1.9635$. Using (39) in (37), for fixed A, we arrive at

$$\frac{1}{2\lambda(A)}\left(\log x + C\right) - Ae^{-\frac{x}{\cosh^2 A}} \leq 2F(x) \leq \frac{1}{2}\log x + \frac{C}{2}. \tag{40}$$

Since A is arbitrary, it can be chosen to grow with x, for example $\frac{1}{2}(1+\epsilon)\log x$, in which case $\lambda(A) \to 1$ and the second term goes to 0, so the upper and lower bounds approach each other. Thus we conclude that as $x \to \infty$,

$$F(x) \to \frac{1}{4}\log x + D, \tag{41}$$

where $D = \frac{C}{4}$. Collecting the results, and remembering that $Q_p(x) = I_1(x) + I_2(x)$, we have

$$Q_p(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ \frac{1}{4}\log x + D & x \to \infty. \end{cases} \tag{42}$$

where $D =\approx 0.49088$, and we have used the fact that $\beta_1 \approx \beta_2$. The asymptotic behavior is illustrated in Figure 3.

3 Discrete Random Walk

We model the Brownian motion as a discrete random walk at the times $t_i = i\Delta t$, where $\Delta t = T/n$ and $i = 0, 1, \ldots, n$. Define $X_i = X(t_i)$, the position of the random walk at time t_i. We assume that X_i has the following dynamics

$$X_{i+1} = \begin{cases} X_i + \delta & \text{with prob } p, \\ X_i - \delta & \text{with prob } q = 1 - p. \end{cases} \tag{43}$$

Defining

$$\delta = \sqrt{\sigma^2 \Delta t + \mu^2 \Delta t^2}, \quad p = \frac{1}{2}\left(1 + \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}}\right), \quad q = \frac{1}{2}\left(1 - \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2 \Delta t}}\right), \tag{44}$$

in the limit $\Delta t \to 0$, the random walk converges in distribution to a Brownian motion on $[0, T]$ with drift μ and diffusion parameter σ.

Analogous to $D(t)$ in Section 2, define D_t to be the drawdown from the previous max at time step t, with $D_0 = 0$. The maximum drawdown is given by $\bar{D} = \max_t D_t$. D_t is a random walk with probability $\tilde{p} = 1 - p$ and a reflective barrier at 0. Let $h > 0$ be an absorbing barrier and let $f(i|h)$ be the probability that the random walk gets absorbed at time step i. Then,

$$P[\bar{D} > h] = P[\text{absorbtion} \in [0, T]] = \sum_{i=0}^{T/\Delta t} f(i|h). \tag{45}$$

$f(i|h)$ was initially computed in [Weesakul, 1961] for $p/q < (1 + 1/N)^2$, the more general case being given in [Blasi, 1976], which after the correction of some typographic errors is given by

$$f(i|h) = \begin{cases} \tilde{f}(1) & \frac{p}{q} < \left(1 + \frac{1}{N}\right)^2, \\ \tilde{f}(2) + \frac{3}{2}\frac{2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)}}{(N+1)(N+\frac{1}{2})} & \frac{p}{q} = \left(1 + \frac{1}{N}\right)^2, \\ \tilde{f}(2) + \frac{2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} q^{\frac{1}{2}} \cosh^{i-1}\beta \sinh^2\beta}{(N+1)q^{\frac{1}{2}}\cosh(N+1)\beta - Np^{\frac{1}{2}}\cosh N\beta} & \frac{p}{q} > \left(1 + \frac{1}{N}\right)^2, \end{cases} \tag{46}$$

where $N = h/\delta$, and

$$\tilde{f}(k) = -2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} \sum_{v=k}^{N} \frac{q^{\frac{1}{2}} \cos^{i-1}\alpha_v \sin^2\alpha_v}{(N+1)q^{\frac{1}{2}}\cos(N+1)\alpha_v - Np^{\frac{1}{2}}\cos N\alpha_v}, \tag{47}$$

and where $\alpha_v \in \left(\frac{v\pi}{N-1}, \frac{(v+1)\pi}{N-1}\right)$ satisfies

$$q^{\frac{1}{2}} \sin(N+1)\alpha_v - p^{\frac{1}{2}} \sin N\alpha_v = 0, \tag{48}$$

and β satisfies

$$q^{\frac{1}{2}} \sinh(N+1)\beta - p^{\frac{1}{2}} \sinh N\beta = 0. \tag{49}$$

Using (46) and (47) in (45) gives the distribution function in the discrete case.

Using δ and p given in (44), and taking the limit $\Delta t \to 0$ gives the continuous case as follows. The sum in (45) is the Riemann sum approximation to the integral $\int_0^T dt\ f_\tau(t|h)$, where $f_\tau(t|h)$ is the absorbtion time density given by the limit of $f(i|h)/\Delta t$ where $i\Delta t = t$. It thus remains to take the $\Delta t \to 0$ limit of $f(i|h)/\Delta t$. Since $p = \frac{1}{2}(1+\lambda)$ where $\lambda \to \frac{\mu\sqrt{\Delta t}}{\sigma}$, and $\delta \to \sigma\sqrt{\Delta t}$, the three cases in (46) reduce to $\mu < \sigma^2/h$, $\mu = \sigma^2/h$, and $\mu > \sigma^2/h$ in the limit, analogous to the three cases in (13). Using the identity $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$, we find that

$$2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} = (1-\lambda^2)^{\frac{i}{2}} \left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{N}{2}} \to e^{-\frac{\mu^2 t}{2\sigma^2}} e^{-\frac{\mu h}{\sigma^2}}. \tag{50}$$

Expanding the eigenvalue condition for α_v in (48) to first order in λ, and using some trigonometric identities gives

$$\tan\left(N+\frac{1}{2}\right)\alpha_v \cos\alpha_v = \frac{2}{\lambda}\sin\frac{\alpha_v}{2}. \tag{51}$$

Since $\alpha_v \in \left(\frac{v\pi}{N-1}, \frac{(v+1)\pi}{N-1}\right)$, let $\theta_v = (N+1/2)\alpha_v$. For fixed v, $\alpha_v \to 0$, so we take the first order expansion in α_v to get

$$\tan\theta_v = \frac{\sigma^2}{\mu h}\theta_v, \tag{52}$$

with $\theta_v \in \left(v\pi\frac{N+\frac{1}{2}}{N-1}, (v+1)\pi\frac{N+\frac{1}{2}}{N-1}\right) \to (v\pi, (v+1)\pi]$. In an identical manner, we can analyse the eigenvalue condition for β to get

$$\tanh\left(N+\frac{1}{2}\right)\beta\cosh\beta = \frac{2}{\lambda}\sinh\frac{\beta}{2}. \tag{53}$$

Defining $\eta = (N+\frac{1}{2})\beta$, and taking the limit, we find

$$\tanh\eta = \frac{\sigma^2}{\mu h}\eta. \tag{54}$$

We now analyse the summand in $\tilde{f}$. Since $\sin\alpha_v \to \theta_v/(N+\frac{1}{2})$, the summand becomes

$$\frac{\theta_v^2 \cos^{i-1}\alpha_v}{(N+\frac{1}{2})^2(N+1)\left[\cos(\theta_v+\frac{1}{2}\alpha_v) - A\cos(\theta_v-\frac{1}{2}\alpha_v)\right]}, \tag{55}$$

where $A = \frac{(1+\lambda)^{1/2}}{(1+\frac{1}{N})(1-\lambda)^{1/2}}$. The identity $\lim\limits_{x\to 0}\cos^{1/x^2} x = e^{-1/2}$ gives that $\cos^{i-1}\alpha_v \to e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}$. Thus, after using some double angle formulae and taking the limit, we have for the summand in $\tilde{f}$

$$\frac{\Delta t\sigma^2}{h^2}\frac{\theta_v^2 e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}}{(1-\frac{\mu h}{\sigma^2})\cos\theta_v - \theta_v\sin\theta_v} = \frac{-\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]}{[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]}, \tag{56}$$

where the second equality follows by use of (52) and some trigonometric identities. Plugging all these results back into (47) and dividing by Δt, we finally arrive at the continuous limit of $\tilde{f}$,

$$\tilde{f}(t) \to e^{-\frac{\mu^2 t}{2\sigma^2}} e^{-\frac{\mu h}{\sigma^2}} \frac{\sigma^2}{h^2}\sum_{v=k}^{\infty}\frac{\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}}{[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]}, \tag{57}$$

where θ_v are the positive solutions to the eigenvalue condition $\tan\theta_v = \sigma^2\theta_v/\mu h$. The remaining two cases are handled in exactly analogous ways. For the second case, $\frac{p}{q} = (1+\frac{1}{N})^2$, the additional term can be computed using (50), and on dividing by Δt we get

$$e^{-\frac{\mu^2 t}{2\sigma^2}}\frac{3\sigma^2}{2eh^2} \qquad\qquad \mu = \frac{\sigma^2}{h}. \tag{58}$$

For the third case, $\frac{p}{q} > (1+\frac{1}{N})^2$, define $\eta = (N+\frac{1}{2})\beta$. The identity $\lim_{x\to 0}\cosh^{1/x^2} x = e^{1/2}$, so we get that $\cosh^{i-1}\beta \to \exp(\frac{\sigma^2\eta^2 t}{2h^2})$. Thus, as with (55), we get for the additional term in (46),

$$\frac{\Delta t\sigma^2}{h^2}\frac{\eta^2 e^{\frac{\sigma^2\eta^2 t}{2h^2}}}{N(1-A)\cosh\eta\cosh\frac{1}{2}\beta - N(1+A)\sinh\eta\sinh\frac{1}{2}\beta}, \tag{59}$$

which upon using manipulations similar to those that led to (56), we finally arrive at

$$\frac{\eta\sinh\eta(\mu^2h^2-\sigma^4\eta^2)}{(\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h)}. \tag{60}$$

Using (50) and dividing by Δt, this additional term then becomes

$$\frac{\sigma^2}{h^2}\frac{(\mu^2h^2-\sigma^4\eta^2)\eta\sinh\eta}{(\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h)}e^{-\frac{\mu^2 t}{2\sigma^2}}e^{-\frac{\mu h}{\sigma^2}}e^{\frac{\sigma^2\eta^2 t}{2h^2}} \qquad \mu > \frac{\sigma^2}{h}. \tag{61}$$

Using (57), (58) and (61) in (46), we arrive at the continuous limit of the discrete time density, in agreement with (12).

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A Expected Value of the High, Low and Range

The high H, low L and the range R are defined by

$$H = \sup_{t \in [0,T]} X(t), \qquad L = \inf_{t \in [0,T]} X(t), \qquad R = H - L. \tag{62}$$

We will derive the expected value of R. We have not been able to find this explicit result in the literature for the general asymmetric Brownian motion, though for the symmetric case, [Feller, 1951] gives the result. Consider the Brownian motion with an absorbing barrier at h. The probability density for the absorbtion time λ is known (see for example [Dominé, 1996]) and is given by the inverse Gaussian distribution

$$f_\lambda(t) = \frac{h}{(2\pi\sigma^2 t^3)^{1/2}} e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{63}$$

Thus, $G_H(h) = P[H \geq h]$ is given by

$$G_H(h) = \int_0^T dt\ f_\lambda(t) = h \int_0^T \frac{dt}{t} \frac{1}{(2\pi\sigma^2 t)^{1/2}} e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{64}$$

The expected value of H is given by $E[H] = \int_0^\infty dh\ G_H(h)$, so we find (after a change of variables to $u = (h - \mu t)/(2\sigma^2 t)^{1/2}$) that

$$E[H] = \frac{1}{\sqrt{\pi}} \int_0^T dt \left(\frac{\sigma^2}{2t}\right)^{1/2} e^{-\alpha^2(t)} + \frac{\mu T}{2} + \frac{\mu}{\sqrt{\pi}} \int_0^T dt \int_0^{\alpha(t)} du\ e^{-u^2}, \tag{65}$$

where $\alpha(t) = \mu t^{1/2}/(2\sigma^2)^{1/2}$. The following identity is useful in evaluating the second integral,

$$\int_0^A dx\; x \int_0^x du\; e^{-u^2} = \frac{\sqrt{\pi}}{4}\mathrm{erf}(A)\left(A^2 - \frac{1}{2}\right) + \frac{A}{4}e^{-A^2},$$

where the error function $\mathrm{erf}(x) = 2/\sqrt{\pi}\int_0^x du\; e^{-u^2}$. The first integral (after a change of variables) can be reduced to $\frac{\sigma^2}{\mu}\mathrm{erf}(\alpha(T))$. The final result is then given by

$$E\left[H\right] = \frac{\mu T}{2} + \frac{\sigma^2}{\mu}\left[\mathrm{erf}(\alpha)\left(\frac{1}{2} + \alpha^2\right) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}}\right], \tag{66}$$

where $\alpha = \alpha(T)$. Expectations for the low and the range can be obtained from the identities $E\left[L\right] = -E\left[H| - \mu\right]$ and $E\left[R\right] = E\left[H|\mu\right] + E\left[H| - \mu\right]$. Using asymptotic forms for $\mathrm{erf}(x)$ [Gradshteyn and Ryzhik, 1980], we find that when $\mu = 0$, $E\left[R\right] = 2\sqrt{2\sigma^2 T/\pi}$, reproducing the result in [Feller, 1951]. Asymptotically, we get

$$E\left[R\right] = \begin{cases} \dfrac{2\sigma^2}{\mu}\left(\dfrac{2\alpha}{\sqrt{\pi}} + \dfrac{2\alpha^3}{\sqrt{\pi}} + \cdots\right) & \alpha \to 0, \\ \dfrac{2\sigma^2}{\mu}\left(\alpha^2 + \dfrac{1}{2} - \dfrac{e^{-\alpha^2}}{\alpha^3} + \cdots\right) & \alpha \to \infty. \end{cases} \tag{67}$$

Thus two different kinds of behavior emerge at the different limits.

B Table of Numerical Values for $Q(\cdot)$

x	$Q_p(x)$, $\mu > 0$		x	$Q_n(x)$, $\mu < 0$
$x \to 0$	$\gamma\sqrt{2x}$		$x \to 0$	$\gamma\sqrt{2x}$
0.0005	0.019690		0.0005	0.019965
0.0010	0.027694		0.0010	0.028394
0.0015	0.033789		0.0015	0.034874
0.0020	0.038896		0.0020	0.040369
0.0025	0.043372		0.0025	0.045256
0.0050	0.060721		0.0050	0.064633
0.0075	0.073808		0.0075	0.079746
0.0100	0.084693		0.0100	0.092708
0.0125	0.094171		0.0125	0.104259
0.0150	0.102651		0.0150	0.114814
0.0175	0.110375		0.0175	0.124608
0.0200	0.117503		0.0200	0.133772
0.0225	0.124142		0.0225	0.142429
0.0250	0.130374		0.0250	0.150739
0.0275	0.136259		0.0275	0.158565
0.0300	0.141842		0.0300	0.166229
0.0325	0.147162		0.0325	0.173756
0.0350	0.152249		0.0350	0.180793
0.0375	0.157127		0.0375	0.187739
0.0400	0.161817		0.0400	0.194489
0.0425	0.166337		0.0425	0.201094
0.0450	0.170702		0.0450	0.207572
0.0500	0.179015		0.0475	0.213877
0.0600	0.194248		0.0500	0.220056
0.0700	0.207999		0.0550	0.231797
0.0800	0.220581		0.0600	0.243374
0.0900	0.232212		0.0650	0.254585
0.1000	0.243050		0.0700	0.265472
0.2000	0.325071		0.0750	0.276070
0.3000	0.382016		0.0800	0.286406
0.4000	0.426452		0.0850	0.296507
0.5000	0.463159		0.0900	0.306393
1.5000	0.668992		0.0950	0.316066
2.5000	0.775976		0.1000	0.325586
3.5000	0.849298		0.1500	0.413136
4.5000	0.905305		0.2000	0.491599
10.0000	1.088998		0.2500	0.564333
20.0000	1.253794		0.3000	0.633007
30.0000	1.351794		0.3500	0.698849
40.0000	1.421860		0.4000	0.762455
50.0000	1.476457		0.5000	0.884593
150.0000	1.747485		1.0000	1.445520
250.0000	1.874323		1.5000	1.970740
350.0000	1.958037		2.0000	2.483960
450.0000	2.020630		2.5000	2.990940
1000.0000	2.219765		3.0000	3.492520
2000.0000	2.392826		3.5000	3.995190
3000.0000	2.494109		4.0000	4.492380
4000.0000	2.565985		4.5000	4.990430
5000.0000	2.621743		5.0000	5.498820
$x \to \infty$	$\frac{1}{4}\log x + 0.49088$		$x \to \infty$	$x + \frac{1}{2}$

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