A Diophantine equation
(corrected slides)

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In Vassil Dimitrov's talk this afternoon, the following statement was presented as a conjecture.

**Theorem**

*For all non-negative integers* $a, b, c, d, e, f$ *we have*

$$\pm 2^a 3^b \pm 2^c 3^d \pm 2^e 3^f \neq 4985.$$
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\pm 2^a 3^b \pm 2^c 3^d \pm 2^e 3^f \neq 4985.
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The theorem is equivalent to:

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Proof.

Let \( n = \gcd(2^{180} - 1, 3^{180} - 1) \)
\[ = 439564261361225 \]
\[ = 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 181 \]

Claim: For all integers \( a, b, c, d, e, f \) we have

\[ \pm 1 \pm 2^c3^d \pm 2^e3^f \not\equiv 4985 \mod n \]
\[ \pm 2^a \pm 3^d \pm 2^e3^f \not\equiv 4985 \mod n \]
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This is a finite computation, and so is trivial.
No, really! It’s not hard.

Let

\[ S = \{ \pm 2^a \cdot 3^b : a, b \in \mathbb{Z} \} \subset (\mathbb{Z}/n\mathbb{Z}) \]

\[ T = \{ \pm 2^a \pm 3^b : a, b \in \mathbb{Z} \} \subset (\mathbb{Z}/n\mathbb{Z}) \]

Then \# \, S = 64800 and \# \, T = 129543.

For \, t \in \{1, -1\} compute the intersection \, S \cap \{s + t + 4985 : s \in S\},

and also compute the intersection \, S \cap \{t + 4985 : t \in T\}.

All three are empty.

This proves the claim.

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